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Critical Cases for Neutral Functional Differential Equations*

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1. INTRODUCTION

A neutral functional differential equation as defined below includes the scalar differential-difference equation

$$\frac{d}{dt}[x(t) + ax(t-1) + G(x(t-1))] = bx(t) + cx(t-1) + F(x(t), x(t-1)), \tag{1.1}$$

where a, b, c are constants and $G(x), F(y, x)$ are continuous functions of x, y. For any continuous function φ defined on $[-1, 0]$, a solution of (1.1) through φ is a continuous function x defined on some interval $[-1, \alpha)$, $\alpha > 0$, which coincides with φ on $[-1, 0]$ and is such that the expression

$$x(t) + ax(t-1) + G(x(t-1))$$

[not $x(t)$] is continuously differentiable on $(0, \alpha)$ and satisfies (1.1) on $(0, \alpha)$.

It has been shown in [1] that the solution $x = 0$ of (1.1) is asymptotically stable provided the solution $x = 0$ of the linear equation

$$\frac{d}{dt}[x(t) + ax(t-1)] = bx(t) + cx(t-1) \tag{1.2}$$

is asymptotically stable and the functions $G(x), F(y, x)$ as well as their first derivatives vanish at $x = y = 0$. Furthermore, the solution $x = 0$ of (1.2) is asymptotically stable if all roots of the characteristic equation

$$\lambda(1 + ae^{-\lambda}) = b + ce^{-\lambda} \tag{1.3}$$

have real parts $\leqslant -\delta < 0$.

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The purpose of this paper is to obtain sufficient conditions for the zero solution of (1.1) to be asymptotically stable even when some roots of (1.3) have zero real parts. Of course, the discussion involves much more general equations, but it is easier to describe the essential ideas for (1.1). A basic hypothesis for (1.1) is that $|a| < 1$ and no roots of (1.3) have positive real parts. This hypothesis eliminates the possibility of a sequence of distinct roots λ_j of (1.3) having $\operatorname{Re}\lambda_j \to 0$ as $j \to \infty$. Suppose P is the finite-dimensional linear subspace of the space C of continuous functions on $[-1, 0]$ which corresponds to all initial values of solutions of (1.2) of the form $p(t)\, e^{\lambda t}$ where p is a polynomial and λ is a root of (1.3) with $\operatorname{Re}\lambda = 0$. If P has dimension d, it is shown that there exist a d-dimensional manifold P^* in C with zero in P^*, and an ordinary differential equation on P^* such that the stability properties of the zero solution of this equation on P^* determine the stability properties of the zero solution of (1.1). Also, constructive methods are given for obtaining this information about P^*. The case of zero roots is discussed in detail and generalizes the paper of Lefschetz [2] for ordinary differential equations. The results about P^* seem to be new even for ordinary differential equations although a partial result of this type appears in Pliss [3]. For retarded equations (that is, $a = 0$, $G = 0$), Shimanov [4, 5] has given some sufficient conditions for the stability of (1.1) in special cases. For neutral equations, the presence of the term G introduces many new dlfficulties in the discussion and, in fact, seems to prevent the use of the converse theorems of Liapunov, a tool systematically employed by Shimanov. The approach used here follows more closely the method of integral manifolds in the spirit of Krylov, Bogoluibov and Mitropolski [6]. A simple example is given at the end of the paper to illustrate the results.

2. Notation and Background

Let E^n be a real or complex n-dimensional linear vector space with norm $|\cdot|$, $r \geqslant 0$ a given real number and C the space of continuous functions mapping $[-r, 0]$ into E^n with $|\varphi| = \sup_{-r\leqslant\theta\leqslant 0} |\varphi(\theta)|$ for $\varphi \in C$. If x is a continuous function taking $[\sigma - r, \sigma + A]$, $A \geqslant 0$, into E^n, then, for each $t \in [\sigma, \sigma + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $-r \leqslant \theta \leqslant 0$. Suppose $\mu(\theta)$, $\eta(\theta)$, are $n \times n$ matrix functions of bounded variation in θ, $-r \leqslant \theta \leqslant 0$, $\varphi \in C$ and define

$$\begin{aligned}
&\text{(a)}\quad L(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\, \varphi(\theta),\\
&\text{(b)}\quad g(\varphi) = \int_{-r}^{0} [d\mu(\theta)]\, \varphi(\theta), \qquad\qquad (2.1)\\
&\text{(c)}\quad D(\varphi) = \varphi(0) - g(\varphi),
\end{aligned}$$

for all φ in C. The functions L and D are continuous linear operators. Also, suppose $G : C \to E^n$, $F : C \to E^n$, G has a continuous first derivative $G'(\varphi)$, and $G(\varphi)$ depends only upon values of $\varphi(\theta)$ for $\theta < 0$; that is, for any $a \in E^n$ and any sequence $\varphi_n \in C$, $\varphi_n(0) = a$, $n = 1, 2, \ldots$, which converges to φ uniformly on compact subsets of $[-r, 0)$, the limit of $G(\varphi_n)$ exists as $n \to \infty$ and $\lim_{n\to\infty} G(\varphi_n) = G(\varphi)$. $G(\varphi)$, also, suppose $G'(\varphi)$ are uniformly continuous on closed bounded sets in C and there exist continuous scalar functions $\gamma(s)$, $q(s)$, $s \geq 0$, $\gamma(0) = q(0) = 0$, such that

$$
\begin{aligned}
&\text{(a)} \quad \left| \int_{-s}^{0} [d_\theta \mu(\theta)]\, \varphi(\theta) \right| \leqslant \gamma(s) \mid \varphi \mid, \\
&\text{(b)} \quad F(0) = G(0) = 0, \\
&\text{(c)} \quad | F(\varphi) - F(\psi) | \leqslant q(\sigma) \mid \varphi - \psi \mid, \\
&\text{(d)} \quad | G(\varphi) - G(\psi) | \leqslant q(\sigma) \mid \varphi - \psi \mid,
\end{aligned}
\tag{2.2}
$$

for $s \geqslant 0$, $\sigma \geqslant 0$ and all φ, ψ in C and, furthermore, $|\varphi|$, $|\psi| \leqslant \sigma$ in (2.2c), (2.2d).

Our main concern throughout this paper is with the autonomous neutral functional differential equation

$$\frac{d}{dt}[D(x_t) - G(x_t)] = L(x_t) + F(x_t). \tag{2.3}$$

A solution $x = x(\varphi)$ of (2.3) through a point φ in C is a continuous function taking $[-r, A)$, $A > 0$, into E^n such that $x_0 = \varphi$ and $D(x_t) - G(x_t)$ is continuously differentiable and satisfies (2.3) for t in $(0, A)$. It is proved in [7, 8] that there is a unique solution $x(\varphi)$ through φ and $x(\varphi)(t)$ is continuous in (t, φ).

Along with (2.3), we consider the linear system

$$\frac{d}{dt} D(y_t) = L(y_t). \tag{2.4}$$

If the transformation $T(t) : C \to C$ is defined by

$$y_t(\varphi) = T(t)\varphi, \tag{2.5}$$

then it is shown in [9] that $\{T(t), t \geqslant 0\}$ is a strongly continuous semigroup of linear operators with the infinitesimal generator $A : \mathscr{D}(A) \to C$, $A\varphi(\theta) = \dot{\varphi}(\theta)$, $-r \leqslant \theta \leqslant 0$,

$$\mathscr{D}(A) = \{\varphi \in C : \dot{\varphi} \in C, \dot{\varphi}(0) = g(\dot{\varphi}) + L(\varphi)\}, \tag{2.6}$$

and the spectrum $\sigma(A)$ of A consists of those λ which satisfy the characteristic equation

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) = \lambda \left[I - \int_{-r}^{0} e^{\lambda\theta}\, d\mu(\theta)\right] - \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta). \quad (2.7)$$

Let $\{T_D(t), t \geqslant 0\}$ be the strongly continuous semigroup of linear transformations associated with the solution of the equation

$$\frac{d}{dt} D(x_t) = 0. \qquad (2.8)$$

DEFINITION. The *order* a_D of D is defined by

$$a_D = \inf\{\text{real } a : \text{there is a constant } K(a) \text{ with}$$

$$\| T_D(t)\varphi \| \leqslant K(a)e^{at} \| \varphi \|, t \geqslant 0, \varphi \in C, D(\varphi) = 0\}. \qquad (2.9)$$

If

$$D(\varphi) = \varphi(0) - \sum_{k=1}^{N} A_k \varphi(-\tau_k) \qquad (2.10)$$

the A_k are $n \times n$ constant matrices, each $\tau_j > 0$ and τ_j/τ_k is rational for $N > 1$, $r = \max_k \tau_k$ it is shown in [10] that $a_D = -\infty$ for $r = 0$ and $D\varphi = \varphi(0)$ and, otherwise,

$$a_D = \sup \left\{\operatorname{Re} \lambda : \det \left(I - \sum A_k e^{-\lambda\tau_k}\right) = 0\right\}. \qquad (2.11)$$

Suppose $a_D < 0$ and all roots of (2.7) have nonpositive real parts. If $\Lambda = \{\lambda : \det \Delta(\lambda) = 0, \operatorname{Re} \lambda = 0\}$, then Λ is a finite set and it follows from [9] that the space C can be *decomposed by* Λ *as* $C = P \oplus Q$ where P, Q are subspaces of C invariant under $T(t)$ and A, the space P is finite dimensional and corresponds to the initial values of all those solutions of (2.4) which are of the form $p(t)\, e^{\lambda t}$, where $p(t)$ is a polynomial in t and $\lambda \in \Lambda$.

Let $X(t)$ be the $n \times n$ matrix function defined for all $t \in [0, \infty)$, of bounded variation in t and continuous in t from the right such that

$$D(X_t) = \int_0^t L(X_s)\, ds + I, \qquad t \geqslant 0,$$
$$X_0(\theta) = \begin{cases} 0 & -r \leqslant \theta < 0 \\ I & \theta = 0 \end{cases}. \qquad (2.12)$$

Since $X(t)$ is a solution (2.4), it is reasonable to let

$$X_t = T(t)\, X_0. \qquad (2.13)$$

Using the same arguments as in [1, 9], it is easily shown that the solution of (2.3) with initial value φ satisfies the equation

$$x_t - X_0 G(x_t) = T(t)[\varphi - X_0 G(\varphi)] + \int_0^t \{d_s[-T(t-s)X_0]\, G(x_s) + T(t-s)\, X_0 F(x_s)\, ds\}, \quad t \geqslant 0. \tag{2.14}$$

Conversely, any solution of (2.14) satisfies (2.3). The integrals in (2.14) are evaluated at each θ in $[-r, 0]$ as ordinary integrals in E^n. Also, if C is decomposed by Λ as $C = P \oplus Q$, then Eq. (2.14) is equivalent to

$$\begin{aligned}
&\text{(a)}\quad x_t^P - X_0^P G(x_t) = T(t)[\varphi^P - X_0^P G(\varphi)] \\
&\qquad\qquad + \int_0^t \{d_s[-T(t-s)X_0^P]\, G(x_s) + T(t-s)\, X_0^P F(x_s)\, ds\}, \\
&\text{(b)}\quad x_t^Q - X_0^Q G(x_t) = T(t)[\varphi^Q - X_0^Q G(\varphi)] \\
&\qquad\qquad + \int_0^t \{d_s[-T(t-s)X_0^Q]\, G(x_s) + T(t-s)\, X_0^Q F(x_s)\, ds\},
\end{aligned} \tag{2.15}$$

where the superscripts P and Q designate the projections of the corresponding functions onto the subspaces P and Q, respectively. Everything is clear in (2.15) except for the meaning of the projections X_0^P, X_0^Q since X_0 is not continuous. These terms will be defined after we have given an explicit way for determining the projections of C onto P and Q.

Projection operators taking C onto P and Q are easily determined by means of the adjoint differential equation

$$\frac{d}{d\tau}\left[z(\tau) - \int_{-r}^0 z(\tau - \theta)\, d\mu(\theta)\right] = -\int_{-r}^0 z(\tau - \theta)\, d\eta(\theta), \tag{2.16}$$

and the bilinear form

$$\begin{aligned}
(\alpha, \varphi) = \alpha(0)\, D(\varphi) &+ \int_{-r}^0 \int_0^\theta \dot{\alpha}(\xi - \theta)[d\mu(\theta)]\, \varphi(\xi)\, d\xi \\
&- \int_{-r}^0 \int_0^\theta \alpha(\xi - \theta)[d\eta(\theta)]\, \varphi(\xi)\, d\xi
\end{aligned} \tag{2.17}$$

defined for all $\alpha \in C^* = C([0, r], E^n)$, $\dot{\alpha} \in C^*$, $\varphi \in C$.

If $\Phi = (\varphi_1, \ldots, \varphi_\nu)$ is a basis for the initial values of those solutions of (2.4) of the form $p(t)\, e^{\lambda t}$, where p is a polynomial and $\lambda \in \Lambda$ and $\Psi = \operatorname{col}(\psi_1, \ldots, \psi_\nu)$ is a basis for the initial values of those solutions of (2.16) of the form $p(t)\, e^{-\lambda t}$, p a polynomial, $\lambda \in \Lambda$, then it is shown in [9] that the

$\nu \times \nu$ matrix $(\Psi, \Phi) = ((\psi_i, \varphi_j),\ i, j = 1, 2,..., \nu)$ is nonsingular and, therefore, can be assumed to be the identity. If Φ, Ψ are defined in this way, then, for any $\varphi \in C$, we define φ^P, φ^Q by

$$\begin{aligned} \varphi &= \varphi^P + \varphi^Q, \\ \varphi^P &= \Phi(\Psi, \varphi). \end{aligned} \tag{2.18}$$

One can now show that (Ψ, X_0) is well-defined and $(\Psi, X_0) = \Psi(0)$. Therefore, if we put

$$X_0^P = \Phi\Psi(0), \qquad X_0^Q = X_0 - X_0^P, \tag{2.19}$$

the quantities in (2.15) are well-defined.

It is shown in [10] that the hypothesis $a_D < 0$ implies there are $K \geqslant 1$, $\alpha > 0$, such that

$$\begin{aligned} &\text{(a)} \quad | T(t)\varphi | \leqslant Ke^{-\alpha t} | \varphi |, \qquad t \geqslant 0, \varphi \in Q, \\ &\text{(b)} \quad | T(t)X_0^Q | + \int_0^1 | d_sT(t-s)X_0^Q | \leqslant Ke^{-\alpha t}, \qquad t \geqslant 0. \end{aligned} \tag{2.20}$$

3. Integral Manifolds in Critical Cases

Throughout the remainder of this paper, we assume D is given in (2.1), $a_D < 0$ and the space C is decomposed by $\Lambda = \{\lambda : \det \Delta(\lambda) = 0, \operatorname{Re} \lambda = 0\}$ as $C = P \oplus Q$ where P, Q are defined as in the previous section. Our objective in this section is to prove there is a ν-dimensional manifold in C which is an integral manifold of system (2.4) in a neighborhood of zero and the stability properties of the solution $x = 0$ relative to this manifold determine the stability properties of the solution $x = 0$ of (2.4). We remark that if there are some roots of (2.8) with positive real parts, then one could obtain the existence of the manifold corresponding to the roots with zero real parts by using essentially the same procedure as below. It only complicates the notation by forcing one to consider some integrals from 0 to $+\infty$ to take care of the roots with positive real parts.

The form of the variation of constants formula (2.14) suggests the possibility of introducing a new variable for the expression on the left hand side. However, some care must be exercised because the new variable would not be a continuous function on $[-r, 0]$. Let PC be the set of functions φ: $[-r, 0] \to E^n$ which are continuous on $[-r, 0)$, $\varphi(0-)$ exists and let $| \varphi | = | \varphi(0)| + \sup\{| \varphi(\theta)|, -r \leqslant \theta < 0\}$. The space PC is then a Banach space with this norm. Let PC_G be the closed set in PC defined by

$$PC_G = \{\varphi \in PC : \varphi(0) = \varphi(0-) - X_0G(\varphi)\}.$$

Since $G(\varphi)$ does not depend on $\varphi(0)$, the maps

$$\begin{aligned} h : C \to PC_G, \qquad & h(\psi) = \psi - X_0 G(\psi) \\ H : PC_G \to C, \qquad & H(\varphi) = \varphi + X_0 G(\varphi) \end{aligned} \tag{3.1}$$

are homeomorphism and $h \cdot H = H \cdot h$ is the identity.

For each φ in PC_G, there is a unique function $\psi \in C$ such that

$$\varphi = \psi - X_0 G(\psi)$$

and, conversely. Therefore, the semigroup $T(t)$ is well defined on PC_G since it is well defined on C and X_0. On the other hand, $T(t)\varphi$ may not belong to PC_G for $\varphi \in PC_G$. In the following, we let $| T(t)\varphi |$ designate $\sup\{| T(t)\varphi(\theta)|, -r \leqslant \theta \leqslant 0\}$ for $\varphi \in PC_G$.

The domain of definition of the infinitesimal generator A of the semigroup $T(t)$ can be extended to PC by letting $\mathscr{D}(A) = \{\varphi \in PC : \dot\varphi \in PC\}$ and

$$A\varphi(\theta) = \begin{cases} \dot\varphi(\theta), & -r \leqslant \theta < 0, \\ g(\dot\varphi) + L(\varphi), & \theta = 0, \end{cases} \tag{3.2}$$

The variation of constants formula (2.14) for the solution of (2.3) suggests the change of variables $x_t - X_0 G(x_t) = z_t$ to obtain a new equation for z_t in PC_G. This transformation is a well-defined transformation from C to PC_G and preserves stability properties since $G(x_t)$ depends only upon the values of $x_t(\theta)$ for $-r \leqslant \theta < 0$ and, therefore, $G(x_t) = G(z_t)$. Thus, if, in (2.14),

$$\begin{aligned} z_t &= x_t - X_0 G(x_t), \\ x_t &= z_t + X_0 G(z_t) \overset{\text{def}}{=} H(z_t), \end{aligned} \tag{3.3}$$

then (2.14) becomes

$$z_t = T(t)z_0 + \int_0^t \{[-d_s T(t-s)X_0]\, G(z_s) + T(t-s)\, X_0 F(H(z_s))\, ds\}. \tag{3.4}$$

Let Φ, Ψ be the matrices defined in Section 2 for the decomposition $C = P \oplus Q$, $(\Psi, \Phi) = I$, and let E be the $\nu \times \nu$ matrix such that $T(t)\Phi = \Phi \exp Et$, $t \in (-\infty, \infty)$. The spectrum of E is Λ. For any $\varphi \in PC$, one can define (Ψ, φ) and, therefore, it is meaningful to put

$$\varphi^P = \Phi(\Psi, \varphi), \qquad \varphi^Q = \varphi - \varphi^P, \qquad \varphi \in PC. \tag{3.5}$$

In a fixed neighborhood of zero, we may assume the semigroup $T(t)$ satisfies (2.20). In the following, it is assumed that all considered neighborhoods of zero are contained in this fixed neighborhood. Also, Eq. (3.4) can be split as

(2.15) with appropriate substitutions from (3.3). Furthermore, if $z_t^P = \Phi u(t)$, then it follows from (2.15a) and the transformation (3.3) that

$$\Phi u(t) = \Phi e^{Et}u(0) + \Phi \int_0^t \{[-d_s e^{E(t-s)}\Psi(0)]\, G(z_s) + e^{E(t-s)}\Psi(0)\, F(H(z_s))\, ds\}.$$

Therefore, we see that Eq. (3.4) is equivalent to

$$\begin{aligned} z_t &= \Phi u(t) + w_t, \qquad w_t \in Q, \\ \frac{du(t)}{dt} &= Eu(t) + F_1(u(t), w_t)), \\ w_t &= T(t)w_0 + \int_0^t \{-[d_s T(t-s)X_0^Q]\, G_0(u(s), w_s) \\ &\qquad + T(t-s)\, X_0^Q F_0(u(s), w_s)\, ds\}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} F_1(u, \varphi) &= \Psi(0)\, F(H(\Phi u + \varphi)) + E\Psi(0)\, G(\Phi u + \varphi), \\ G_0(u, \varphi) &= G(\Phi u + \varphi), \\ F_0(u, \varphi) &= F(H(\Phi u + \varphi)),\ u \in E^\nu,\ \varphi \in Q. \end{aligned} \tag{3.7}$$

For any $\rho > 0$, let $\Omega_\rho = \{(u, \varphi) \in E^\nu \times Q : 0 \leqslant |u| < \infty, 0 \leqslant |\varphi| \leqslant \rho\}$ and let F_1^e, F_0^e, G_0^e be functions defined on Ω_ρ, which coincide with F_1, F_0, G_0, respectively, on $\{(u, \varphi) \in E^\nu \times Q : 0 \leqslant |u| \leqslant \rho,\ |\varphi| \leqslant \rho\}$ and

$$\begin{aligned} F_j^e(u, \varphi) &= F_j\left(\frac{u\rho}{|u|}, \varphi\right), \qquad j = 0, 1 \\ G_0^e(u, \varphi) &= G_0\left(\frac{u\rho}{|u|}, \varphi\right), \qquad \rho < |u| < \infty. \end{aligned} \tag{3.8}$$

From the properties of F_1, F_0 and G_0, there is a nondecreasing continuous function $\nu(\rho)$, $\rho \geqslant 0$, $\nu(0) = 0$, such that

$$\begin{gathered} F_j^e(0, 0) = 0, \qquad |F_j^e(u, \varphi)| \leqslant \nu(\rho)\rho, \\ |F_j^e(u, \varphi) - F_j^e(v, \psi)| \leqslant \nu(\rho)(|u - v| + |\varphi - \psi|), \qquad j = 0, 1, \\ G_0^e(0, 0) = 0, \qquad |G_0^e(u, \varphi)| \leqslant \nu(\rho)\rho, \\ |G_0^e(u, \varphi) - G_0^e(v, \psi)| \leqslant \nu(\rho)(|u - v| + |\varphi - \psi|), \end{gathered} \tag{3.9}$$

for $(u, \varphi), (v, \psi) \in \Omega_\rho$ (see, e.g., Chafee [11]).

In order to discuss the local properties of (3.6) near $u = 0$, $w_t = 0$, it is convenient to first discuss the global properties of the system

$$\begin{aligned} \frac{du(t)}{dt} &= Eu(t) + F_1{}^e(u(t), w_t), \\ w_t &= T(t)w_0 - \int_0^t [d_sT(t-s)X_0{}^Q]\, G_0{}^e(u(s), w_s) \\ &\quad + \int_0^t T(t-s)\, X_0{}^Q F_0{}^e(u(s), w_s)\, ds, \qquad t \geqslant 0. \end{aligned} \tag{3.10}$$

THEOREM 3.1. *There is a* $\rho_0 > 0$ *and a Lipschitz continuous function* $h : E^\nu \to Q$ *such that for any* $0 < \rho \leqslant \rho_0$, $(u, h(u)) \in \Omega_\rho$, $0 \leqslant |u| < \infty$ *and the set* $M_\rho = \{(u, \varphi) \in \Omega_\rho : \varphi = h(u),\ 0 \leqslant |u| < \infty\}$ *is an integral manifold of* (3.10). *Furthermore, any solution of* (3.10) *with initial value in* M_ρ *is defined for all* $t \in (-\infty, \infty)$.

Proof. For simplicity in the estimates we assume $|\exp Et| = 1$. Only slight changes in the proof are needed since for any $\epsilon > 0$ there is a $K(\epsilon)$ so that $|\exp Et| \leqslant K(\epsilon) \exp \epsilon t$ for all t. With K, α as in (2.20) and $\nu(\rho)$ as in (3.9), for any ρ with $\nu(\rho) = \alpha/4$, there is a constant $K_1 = K_1(\alpha)$ such that

$$\int_{-\infty}^0 |d_sT(-s)X_0{}^Q|\, e^{-2\nu(\rho)s} \leqslant K_1 .$$

In fact, if we write this integral as an infinite sum of integrals of length one and use (2.20), then

$$\begin{aligned} \int_{-\infty}^0 |d_sT(-s)X_0{}^Q|\, e^{-2\nu(\rho)s} &= \sum_{j=0}^{\infty} \int_0^1 |d_sT(j+1-s)X_0{}^Q|\, e^{2\nu(\rho)(j+1-s)} \\ &\leqslant \sum_{j=0}^{\infty} e^{2\nu(\rho)(j+1)} \int_0^1 |d_sT(j+1-s)X_0{}^Q| \\ &\leqslant \sum_{j=0}^{\infty} e^{-(\alpha-2\nu(\rho))(j+1)} \\ &\leqslant \sum_{j=0}^{\infty} e^{-\alpha(j+1)/2} \overset{\text{def}}{=} K_1(\alpha). \end{aligned}$$

Let $K_2 = \max(K_1, K)$, choose ρ_0 so that

$$8K_2(1 + \alpha^{-1})\, \nu(\rho_0) < 1, \tag{3.11}$$

and, for any $0 < \rho \leqslant \rho_0$, let

$$S = \{h : E^\nu \to Q, \text{ continuous}, (u, h(u)) \in \Omega_\rho, u \in E^\nu, h(0) = 0,$$
$$| h(u) - h(v) | \leqslant | u - v |, u, v \in E^\nu\} \tag{3.12}$$

For $h \in S$, let $| h | = \sup\{| h(u)|, u \in E^\nu\}$.

For any $h \in S$, let $u(t) = u(t, u_0, h)$, $u(0, u_0, h) = u_0$, be the solution of the system

$$\dot{u}(t) = Eu(t) + F_1{}^e(u(t), h(u(t))), \qquad t \leqslant 0, \tag{3.13}$$

and define the mapping $\mathscr{P} : S \to$ functions from E^ν into Q by the relation

$$(\mathscr{P}h)(u_0) = -\int_{-\infty}^0 [d_s T(-s) X_0{}^Q]\, G_0{}^e(u(s), h(u(s)))$$
$$+ \int_{-\infty}^0 T(-s)\, X_0{}^Q F_0{}^e(u(s), h(u(s)))\, ds, \tag{3.14}$$

for $u_0 \in E^\nu$. Our objective is to show that $\mathscr{P}$ has a fixed point in S and then to show that this fixed point defines an integral manifold M_ρ.

For any h in S, it follows from (3.9), (3.11) and our estimates on $T(t)$ in (2.20) that

$$|(\mathscr{P}h)(u_0)| \leqslant K_2(\rho)\, \rho(1 + \alpha^{-1}) \leqslant \rho.$$

To estimate the dependence of $(\mathscr{P}h)(u_0)$ upon h and u_0 we need the dependence of $u(t, u_0, h)$ upon the same parameters. From (3.13), one easily obtains from the variation of constants formula and simple estimates that

$$| u(t, u_0, h) - u(t, \bar{u}_0, \bar{h})| \leqslant e^{-2\nu(\rho)t}(| u_0 - \bar{u}_0 | + \tfrac{1}{2} | h - \bar{h} |), \qquad t \leqslant 0.$$

Since

$$| G_0{}^e(u(t, u_0, h), h(u(t, u_0, h))) - G_0{}^e(u(t, \bar{u}_0, \bar{h}), \bar{h}(u(t, \bar{u}_0, \bar{h})))|$$
$$\leqslant 2\nu(\rho) | u(t, u_0, h) - u(t, \bar{u}_0, \bar{h})| + \nu(\rho) | h - \bar{h} |,$$

it follows that

$$\Big| \int_{-\infty}^{0} [d_s T(-s) X_0^Q][G_0^e(u(s, u_0, h), h(u(s, u_0, h)))$$

$$- G_0^e(u(s, \bar{u}_0, \bar{h}), \bar{h}(u(s, \bar{u}_0, \bar{h})))] \Big|$$

$$\leqslant \int_{-\infty}^{0} | d_s T(-s) X_0^Q | \, [2\nu(\rho) e^{-2\nu(\rho)s}(| u_0 - \bar{u}_0 | + \tfrac{1}{2} | h - \bar{h} |) + \nu(\rho) | h - \bar{h} |]$$

$$\leqslant K\nu(\rho) | h - \bar{h} | + K_1 2\nu(\rho)(| u_0 - \bar{u}_0 | + \tfrac{1}{2} | h - \bar{h} |)$$

$$\leqslant 2\nu(\rho) K_2(| u_0 - \bar{u}_0 | + | h - \bar{h} |).$$

In a similar manner, one shows that

$$\Big| \int_{-\infty}^{0} T(-s)[F_0^e(u(s, u_0, h), h(u(s, u_0, h))) - F_0^e(u(s, \bar{u}_0, \bar{h}), \bar{h}(u, s, \bar{u}_0, \bar{h})))] \, ds \Big|$$

$$\leqslant \frac{4K\nu(\rho)}{\alpha} (| u_0 - \bar{u}_0 | + \tfrac{1}{2} | h - \bar{h} |) + \frac{K\nu(\rho)}{\alpha} | h - \bar{h} |$$

$$\leqslant \frac{4K_2\nu(\rho)}{\alpha} (| u_0 - \bar{u}_0 | + | h - \bar{h} |).$$

Combining these estimates and using (3.11), we obtain

$$|(\mathscr{P}h)(u_0) - (\mathscr{P}\bar{h})(\bar{u}_0)| \leqslant | u_0 - \bar{u}_0 | + \tfrac{1}{2} | h - \bar{h} |.$$

Therefore, $\mathscr{P} : S \to S$ and is a contraction. The unique fixed point h of $\mathscr{P}$ in S is easily shown to define an integral manifold satisfying the properties of the theorem. This completes the proof.

Our next objective is to determine the stability properties of the manifold M_ρ given in Theorem 3.1. To do this, we need

LEMMA 3.1. *There are* $\rho_1 > 0$, $\rho_2 < 0$, $K_3 > 0$, $\alpha_1 > 0$, *and a continuous function* $p : R^+ \times \Omega_{\rho_2} \to Q$ *such that if* $(u(t), w_t)$ *is a solution of* (3.10) *with initial value* $(u_0, \varphi) \in \Omega_{\rho_2}$, *then* $(u(t), w_t)$ *exists for* $t \geqslant 0$ *and*

$$w_t = p(t, u(t), \varphi), \qquad t \geqslant 0. \tag{3.15}$$

Moreover, p *satisfies the inequalities*

$$\begin{aligned} &\text{(a)} \quad | p(t, u, \varphi) | \leqslant \rho_1, \\ &\text{(b)} \quad | p(t, u, \varphi) - p(t, \bar{u}, \bar{\varphi}) | \leqslant | u - \bar{u} | + K_3 e^{-\alpha_1 t} | \varphi - \bar{\varphi} |, \end{aligned} \tag{3.16}$$

for $(t, u, \varphi), (t, \bar{u}, \bar{\varphi}) \in R^+ \times \Omega_{\rho_2}$.

Proof. Consider the equation

$$\frac{du(\tau)}{d\tau} = Eu(\tau) + F_1^e(u(\tau), p(\tau, u(\tau), \varphi)), \qquad 0 \leqslant \tau \leqslant t,$$

$$u(t) = u_0 ,$$

$$p(t, u_0, \varphi) = T(t)\varphi - \int_0^t [d_s T(t-s) X_0^Q]\, G_0^e(u(s), p(s, u(s), \varphi))$$

$$+ \int_0^t T(t-s)\, X_0^Q F_0^e(u(s), p(s, u(s), \varphi))\, ds,$$

for $(t, u_0, \varphi) \in R^+ \times \Omega_{\rho_0}$. By using arguments very similar to the proof of Theorem 3.1, one proves the existence of a $p(t, u, \varphi)$ satisfying (3.16a) which is Lipschitzian in u, φ. To show the precise estimate (3.16b) is more difficult and one can use an argument similar to the one used in the basic stability theorem in [1] to obtain this estimate. It is then easy to verify (3.15). The details are omitted.

THEOREM 3.2. *With h as in Theorem 3.1 there are constants $K_1 > 0$, $\rho_3 > 0$ such that any solution $(u(t), w_t)$ of (3.10) with initial value $(u_0, \varphi) \in \Omega_{\rho_3}$, is defined for all $t \geqslant 0$ and satisfies*

$$| w_t - h(u(t))| \leqslant K_1 e^{-\alpha_1 t} | \varphi - h(u_0)|, \qquad t \geqslant 0.$$

Proof. If M_{ρ_2} is the integral manifold in Theorem 3.1, then any solution lying in M_{ρ_2} must satisfy (3.15). If $(u(t), w_t)$ is any solution of (3.10) with initial value in Ω_{ρ_2}, then Lemma 3.1 implies that this solution is defined for all $t \geqslant 0$. For an arbitrary $\tau \geqslant 0$, the solution of (3.10) through $(u(\tau), h(u(\tau))$ is defined for all $t \in (-\infty, \infty)$ and lies on M_{ρ_2} from Theorem 3.1. This solution can therefore be considered as the value of a solution of (3.10) at time τ starting from some point (u^*, φ^*) at $\tau = 0$. Lemma 3.1, therefore, implies

$$\begin{aligned} | w_\tau - h(u(\tau))| &= | p(\tau, u(\tau), \varphi) - p(\tau, u(\tau), \varphi^*)| \\ &\leqslant K_3 e^{-\alpha_1 \tau} | \varphi - \varphi^* | = K_3 e^{-\alpha_1 \tau} | \varphi - h(u^*)| \\ &\leqslant K_3 e^{-\alpha_1 \tau} [| \varphi - h(u_0)| + | h(u_0) - h(u^*)|] \end{aligned}$$

for $\tau \geqslant 0$.

To estimate $h(u_0) - h(u^*)$, it is sufficient to estimate $| u_0 - u^* |$ since h is Lipschitzian and, in fact, the proof of Theorem 3.1 showed the Lipschitz constant could be chosen equal to one. The quantities u_0, u^* correspond to initial values, respectively, of solutions of

$$\frac{du(s)}{ds} = Eu(s) + F_1^e(u(s), p(s, u(s), \varphi)),$$

$$\frac{du^*(s)}{ds} = Eu^*(s) + F_1^e(u^*(s), p(s, u^*(s), \varphi^*)),$$

with $u(\tau) = u^*(\tau)$. Using the variation of the constants formula for the solutions of these equations through $u(\tau), u^*(\tau)$, respectively, and the properties of F_1^e, p, one easily obtains, from Gronwall's inequality,

$$| u^*(s) - u(s)| \leqslant \theta \mid \varphi - \varphi^* \mid, \qquad 0 \leqslant s \leqslant \tau,$$

where $\theta = K_3\nu(\rho_1)/[\alpha_1 - 2\nu(\rho_1)]$. Further restrict ρ_1 so that $\theta < 1$. Then

$$\begin{aligned} | h(u^*) - h(u_0)| &\leqslant | u^* - u_0 | \leqslant \theta \mid \varphi - h(u^*)| \\ &\leqslant \theta[| \varphi - h(u_0)| + | h(u_0) - h(u^*)|] \end{aligned}$$

and

$$| h(u^*) - h(u_0)| \leqslant \frac{1}{1-\theta} \mid \varphi - h(u_0)|.$$

This fact together with the previous estimate on $| w_\tau - h(u(\tau))|$ implies

$$| w_\tau - h(u(\tau))| \leqslant \frac{K_3}{1-\theta} e^{-\alpha_1\tau} \mid \varphi - h(u_0)|$$

and proves the theorem.

COROLLARY 3.1. *With h as in Theorem 3.1 and ρ_3 as in Theorem 3.2, any solution $(u(t), w_t) \in \Omega_{\rho_3}$ for $t \leqslant 0$ must lie on M_{ρ_3}.*

Let h be as in Theorem 3.1 and ρ_3 as in Theorem 3.2 and consider the ordinary differential equation

$$\frac{du(t)}{dt} = Eu(t) + F_1^e(u(t), h(u(t))) \tag{3.17}$$

which describes the behavior of the solutions $(u(t), w_t) = (u(t), h(u(t))$ of (3.10) on the integral manifold M_{ρ_3}.

THEOREM 3.3. *If the solution $u = 0$ of (3.17) is uniformly asymptotically stable (unstable), then the solution $u = 0$, $w_t = 0$ of (3.10) [and therefore the solution $x = 0$ of (2.3)] is uniformly asymptotically stable (unstable).*

Proof. For any $\delta > 0$, let $B_\delta = \{(u, \varphi) \in E^\nu \times Q : |u| + |\varphi| < \delta\}$. Let $M_{\rho_3} \cap B_{b_0}$ be contained in the region of attraction of the solution $u = 0$ of (3.17). We first prove stability. From the hypothesis on (3.17), for any $\epsilon > 0$, $2\epsilon < \rho_3$, there is a $\delta > 0$, $\delta < b_0$, such that any solution $u(t)$ of (3.17) satisfies $|u(t)| < \epsilon$ for $t \geqslant 0$ provided that $|u(0)| < \delta$. Also, there is a $t_0 = t_0(\delta)$ such that $|u(t)| < \delta/2$ for $t \geqslant t_0$ since the zero solution of (3.17) is assumed to be uniformly asymptotically stable. With K_1, α_1 as in Theorem 3.2, further restrict δ so that $K_1 \exp(-\alpha_1 t_0(\delta)) < 1/2$. Such a choice of δ is possible since $t_0(\delta) \to \infty$ as $\delta \to 0$.

From continuous dependence on initial data in (3.10), there is a neighborhood $V_{\delta_1} = \{(u, \varphi) : |u| < \delta + \delta_1, |h(u) - \varphi| < \delta_1\}$ of $M_{\rho_3} \cap B_\delta$ such that $(u(0), w_0) \in V_{\delta_1}$ implies the solution $(u(t), w_t)$ of (3.10) belongs to $B_{2\epsilon}$ for $0 \leqslant t \leqslant t_0$ and $(u(t_0), w_{t_0})$ is in a $\delta/2$ neighborhood $W_{\delta/2}$ of $M_{\rho_3} \cap B_{\delta/2}$. Since $K_1 \exp(-\alpha_1 t_0) < 1/2$, it follows that $|w_{t_0}| < \delta_1/2$ and, thus, $(u(t_0), w_{t_0}) \in V_{\delta_1}$. Therefore, $(u(t), w_t)$ must belong to $B_{2\epsilon}$ for all $t \geqslant 0$. This proves stability of the zero solution of (3.10).

Suppose V_{δ_1} is chosen as above and $(u(0), w_0) \in V_{\delta_1}$. The solution $(u(t), w_t)$ of (3.10) through $(u(0), w_0)$ is in $B_{2\epsilon}$. Since $2\epsilon < \rho_1$, this defines a solution x of (2.3) which is bounded. Since $a_D < 0$, it follows from [12] that the orbit of this solution has a nonempty ω-limit set. Theorem 3.2 implies this limit set must be in $M_{\rho_3} \cap B_\delta$. Since the ω-limit set is invariant and the only invariant set in $M_{\rho_3} \cap B_\delta$ is zero [by the hypothesis on system (3.17)] it follows that every solution of (3.10) with initial value in B_{δ_1} approaches zero as $t \to \infty$. This completes the proof of the asymptotic stability.

If the solution $u = 0$ is unstable, then it is obvious that the zero solution of (3.10) is unstable. This completes the proof of the theorem.

4. Stability in Critical Cases—Zero Roots

In the previous section, we proved a result (Theorem 3.3) which stated that the asymptotic stability (or instability) of the zero solution of (2.3) is determined by the asymptotic stability (or instability) of the zero solution of an ordinary differential equation (3.17). It, therefore, remains to analyze the behavior of the solutions of (3.17) near $u = 0$. Of course, this is an extremely difficult task and no general procedure is available to treat all possible situations. Therefore, one is forced to consider particular cases, one of which will be discussed in this section.

Suppose D, L, G, F as before and $a_D < 0$. Also, suppose $x = 0$ is an isolated equilibrium point of (2.3) and if $\varphi(y)$ is an analytic function of a ν-vector y in a neighborhood of $y = 0$, then $G(\varphi(y)), F(\varphi(y))$ are analytic functions of y in a neighborhood of $y = 0$. Finally, $\lambda = 0$ is a root of the characteristic equation (2.7) of multiplicity ν, the dimension of the null space of $\Delta(0)$ is ν and all other roots of (2.7) have negative real parts; i.e., Λ of Section 2 consists only of the element 0, and if C is decomposed by Λ as $C = P \oplus Q$, then a basis Φ for P can be chosen to consist of constant functions. Since $T(t)\,\Phi = \Phi$, the matrix E in (3.6) is zero. For notational convenience throughout this section, if φ is a constant function in C, then φ_0 will denote the value of this function in E^n.

If Ψ is a basis for the constant solutions of the adjoint equation (2.16) with $(\Psi, \Phi) = I$, then a direct computation shows that

$$\begin{aligned} I &= (\Psi, \Phi) = \Psi_0 \dot{\Delta}(0) \Phi_0\,, \\ \dot{\Delta}(0) &= I - \int_{-r}^{0} d\mu(\theta) - \int_{-r}^{0} \theta\, d\eta(\theta). \end{aligned} \tag{4.1}$$

Suppose PC is the space of functions defined in Section 3 and A is defined by (3.1). Let $\Delta^{\#}(0)$ be that $n \times n$ matrix which takes the range of $\Delta(0)$ onto the range of the transpose $\Delta'(0)$ of $\Delta(0)$ in a one-to-one manner and $\Delta(0)\,\Delta^{\#}(0) = I$ on the range of $\Delta(0)$.

LEMMA 4.1. *If $Q = \{\varphi \in PC : (\Psi, \varphi) = 0\}$ and A is defined by* (3.2), *then there is a bounded right inverse A^{-1} of A taking $PC \cap R(A)$ into $Q \cap \mathscr{D}(A)$; that is, if $\varphi \in PC$, $\psi = A^{-1}\varphi$, then $\psi \in Q^* \cap \mathscr{D}(A)$ and $A\psi = AA^{-1}\varphi = \varphi$. Also, if $\varphi \in PC \cap R(A)$, then $A^{-1}\varphi$ is defined by*

$$\begin{aligned} (A^{-1}\varphi)(\theta) &= \varphi_1(\theta) + \Phi_0 a, \\ \varphi_1(\theta) &= \int_0^{\theta} \varphi(s)\, ds + c_1(\varphi), \qquad -r \leqslant \theta \leqslant 0, \\ c_1(\varphi) &= \Delta^{\#}(0)(I, \varphi), \\ a &= -(\Psi, \varphi_1), \end{aligned} \tag{4.2}$$

where (I, φ) is the bilinear form defined in (2.17) *evaluated at the identity I.*

Proof. Suppose $\varphi \in PC \cap R(A)$. It is not dlfficult to show this implies $(I, \varphi) \in R(\Delta(0))$. With A^{-1} defined in (4.2), it is clear that A^{-1} is a bounded linear operator taking PC in Q^*. Also $dA^{-1}\varphi/d\theta$ is in PC and, for $-r \leqslant \theta < 0$,

$$(AA^{-1}\varphi)(\theta) = \frac{d}{d\theta}(A^{-1}\varphi)(\theta) = \varphi(\theta).$$

Since $g(\varphi)$ satisfies (2.2a), $(AA^{-1}\varphi)(\theta) = d(A^{-1}\varphi)(\theta)/d\theta$, $-r \leqslant \theta < 0$, $L(\Phi) = 0$ and $\Delta(0)\,\Delta^{\#}(0) = I$, we have

$$\begin{aligned} g\left(\frac{d}{d\theta} A^{-1}\varphi\right) + L(A^{-1}\varphi) &= g(\varphi) + L(\varphi_1 + \Phi a) \\ &= g(\varphi) + L(\varphi_1) \\ &= g(\varphi) + L\left(\int_0^\theta \varphi(s)\, ds\right) + f(\Delta^{\#}(0)(I, \varphi)) \\ &= g(\varphi) + \int_{-r}^0 [d\eta(\theta)] \int_0^\theta \varphi(s)\, ds + \Delta(0)\,\Delta^{\#}(0)(I, \varphi) \\ &= \varphi(0). \end{aligned}$$

This completes the proof of the lemma.

Let X_0 be the $n \times n$ matrix function on $[-r, 0]$ defined by $X_0(\theta) = 0$, $-r \leqslant \theta < 0$, $X_0(0) = I$, the identity.

LEMMA 4.2. *Suppose $\tilde{G}, \tilde{F}$ are arbitrary functions satisfying* (2.2b)–(2.2d), *$\tilde{G}(\varphi)$ depends only on the values of $\varphi(\theta)$ for $\theta < 0$, and $\tilde{G}(\varphi(y))$, $\tilde{F}(\varphi(y))$ are analytic functions of the ν-vector y in a neighborhood of $y = 0$ if $\varphi(y)$ is analytic in y in a neighborhood of $y = 0$. With A^{-1} as in Lemma* 4.1, *the equation*

$$\varphi + X_0{}^Q\tilde{G}(\Phi y + \varphi) = -A^{-1}X_0{}^Q\tilde{F}(\Phi y + \varphi) \tag{4.3}$$

has a unique solution $\varphi^Q = \alpha(y)$ in a neighborhood of $y = 0$, $\varphi^Q = 0$, and the solution is analytic in y in a neighborhood of $y = 0$. Furthermore, if the power series expansion of $\tilde{G}(\Phi y + \varphi^Q)$, $\tilde{F}(\Phi y + \varphi^Q)$ begins with the terms of degree k in y, then the power series expansion of $\alpha(y)$ begins with terms of at least degree k in y.

The proof is not given since it is a standard application of the method of successive approximations and the properties of A^{-1}.

We now apply these lemmas to the study of the stability of the zero solution of (2.3). Suppose system (2.3) has been transformed to the system (3.6) through the transformation (3.3). We now make an additional transformation on (3.6) to a more convenient form.

Let $\alpha(u)$ be the solution assured by Lemma 4.2 of the equation

$$\varphi + X_0{}^QG(\Phi u + \varphi) = -A^{-1}X_0{}^QF(H(\Phi u + \varphi)) \tag{4.4}$$

and consider the transformation of variables

$$w_t = v_t + \alpha(u(t)) \tag{4.5}$$

in (3.6). If $\beta(u) = \Phi u + \alpha(u)$, $F \circ H = \tilde{F}$, the new equation for v_t is given by

$$\begin{aligned} v_t = {} & T(t)v_0 + T(t)\,\alpha(u(0)) - \alpha(u(t)) \\ & + \int_0^t \{d_s[-T(t-s)X_0{}^Q]\, G(\beta(u)) + T(t-s)\, X_0{}^Q\tilde{F}(\beta(u))\, ds\} \\ & + \int_0^t d_s[-T(t-s)X_0{}^Q]\{G(\beta(u)+v_s) - G(\beta(u))\} \\ & + \int_0^t T(t-s)\, X_0{}^Q\{\tilde{F}(\beta(u)+v_s) - \tilde{F}(\beta(u))\}\, ds, \end{aligned} \tag{4.6}$$

where the function u under the integrals is always evaluated at s. Since $u(t)$ has a continuous first derivative, the function $G(\Phi u(t) + \alpha(u(t)))$ has a continuous first derivative. Therefore, the first term in the first integral in (4.6) can be integrated by parts to obtain

$$\int_0^t T(t-s)\, X_0{}^Q d_s G(\beta(u(s))) + T(t)\, X_0{}^Q G(\beta(u(0))) - X_0{}^Q G(\beta(u(t))). \tag{4.7}$$

If $\gamma(u(t)) = \alpha(u(t)) + X_0{}^Q G(\beta(u(t)))$, then $\gamma(u(t))$ is in $R(A^{-1})$ for each t and $\gamma(u(t))$ is also continuously differentiable in t. From the definition of A^{-1} in Lemma 4.1, $T(0)\, A^{-1}\varphi - T(t)\, A^{-1}\varphi = -\int_0^t T(t-s)\varphi\, ds$ for all $\varphi \in PC$, $t \geqslant 0$. Using this fact, one obtains

$$\begin{aligned} T(t)\,\gamma(u(0)) - T(0)\,\gamma(u(t)) = {} & + \int_0^t T(t-s)\, A\gamma(u(s))\, ds \\ & - \int_0^t T(t-s)\, \frac{\partial \gamma(u(s))}{\partial s}\, ds. \end{aligned} \tag{4.8}$$

If we use these relations (4.7), (4.8) in (4.6) together with the fact that $\alpha(u)$ satisfies (4.4), we obtain

$$\begin{aligned} v_t = {} & T(t)v_0 - \int_0^t [d_s T(t-s)X_0{}^Q]\{G(\beta(u(s))) + v_s) - G(\beta(u(s)))\} \\ & + \int_0^t T(t-s)\, X_0{}^Q\{\tilde{F}(\beta(u(s)) + v_s) - \tilde{F}(\beta(u(s)))\}\, ds \\ & - \int_0^t T(t-s)\, \frac{\partial \alpha(u(s))}{\partial u}\, \dot{u}(s)\, ds, \\ \beta(u) = {} & \Phi u + \alpha(u), \qquad \tilde{F} = F \circ H, \end{aligned} \tag{4.9}$$

where α satisfies (4.4) and H is defined in (3.3).

We summarize these results in the following

THEOREM 4.1. *Suppose D, L, G, F satisfy the conditions of Section 2, zero is a characteristic value of (2.4) whose geometric and algebraic multiplicities are ν and $G(\varphi(y))$, $F(\varphi(y))$ are analytic functions of the ν-vector y in a neighborhood of $y = 0$. Suppose PC is the space of functions mapping $[-r, 0]$ into E^n defined in Section 3 and $A : \mathscr{D}(A) \to PC$, $\mathscr{D}(A) \subset PC$ is defined by (3.2). Let C be decomposed by $\{0\}$ as $C = P \oplus Q$. With H defined as in (3.3), let $\alpha(u)$ be the solution of (4.4). If*

$$\begin{aligned} &\text{(a)}\quad x_t = H(z_t),\ \beta(u) = \Phi u + \alpha(u), \\ &\text{(b)}\quad z_t = \beta(u(t)) + v_t,\ v_t \in Q, \\ &\text{(c)}\quad \hat{F}(u, \varphi) = F(H(\beta(u) + \varphi)), \\ &\text{(d)}\quad \hat{G}(u, \varphi) = G(\beta(u) + \varphi), \end{aligned} \tag{4.10}$$

then the initial problem for (2.3) in a neighborhood of $\varphi = 0$ is equivalent to the equations

$$\begin{aligned} &\text{(a)}\quad \frac{du(t)}{dt} = \Psi(0)\,\hat{F}(u(t), v_t), \\ &\text{(b)}\quad v_t = T(t)v_0 - \int_0^t [d_sT(t-s)X_0{}^Q][\hat{G}(u(s), v_s) - \hat{G}(u(s), 0)] \\ &\qquad + \int_0^t T(t-s)\,X_0{}^Q[\hat{F}(u(s), v_s) - \hat{F}(u(s), 0)]\,ds \\ &\qquad - \int_0^t T(t-s)\,\frac{\partial\alpha(u(s))}{\partial u}\,\Psi(0)\,\hat{F}(u(s), v_s)\,ds. \end{aligned} \tag{4.11}$$

There is a degenerate case of (4.11) which needs to be discussed separately; viz., the case in which F in (2.3) satisfies $F(\varphi) = 0$ for all φ in a neighborhood of $\varphi = 0$. Equations (4.11) for this case are

$$\frac{du(t)}{dt} = 0,$$

$$v_t = T(t)v_0 - \int_0^t [d_sT(t-s)X_0{}^Q][\hat{G}(u(s), v_s) - \hat{G}(u(s), 0)].$$

Using the same type of argument as in the proof of the stability theorem in [1], one can show that the solution $(u, v_t) = (0, 0)$ is uniformly stable and, thus, the solution $x = 0$ of (2.3) is uniformly stable. That is, a perturbation in (2.3) which occurs only in the term which is being differentiated does not affect the zero roots of (2.7) provided that the algebraic and geometric multiplicity of this root are the same.

We now discuss the case when $F(\varphi) \not\equiv 0$. System (4.11) is of the same form as (3.6). The results in Section 3 did not depend upon the form of F_1, F_0, G_0 but only upon the estimates (3.9). Therefore, the conclusion of Theorem 3.3 is valid for our situation and it remains only to analyze the behavior of the solutions of (3.17) under our present hypotheses. Now the form of the terms in (4.11) are important since they permit us to determine the qualitative behavior of the integral manifold given in Theorem 3.1 near $u = 0$.

To be more specific, let us define

$$\begin{aligned} F_1(u, \varphi) &= \Psi_0 \hat{F}(u, \varphi), \\ G_0(u, \varphi) &= \hat{G}(u, \varphi) - \hat{G}(u, 0), \\ F_0(u, \varphi) &= X_0{}^Q[\hat{F}(u, \varphi) - \hat{F}(u, 0)] + \frac{\partial \alpha(u)}{\partial u} \Psi_0 \hat{F}(u, \varphi), \end{aligned} \tag{4.12}$$

and write (4.11) as

$$\begin{aligned} \frac{du(t)}{dt} &= F_1(u, v_t) \\ v_t &= T(t)v_0 - \int_0^t [d_s T(t-s) X_0{}^Q]\, G_0(u(s), v_s) \\ &\quad + \int_0^t T(t-s)\, F_0(u(s), v_s)\, ds. \end{aligned} \tag{4.13}$$

If the functions F_1, G_0, F_0 in (4.12) are extended as $F_1{}^e$, $G_0{}^e$, $F_0{}^e$ in (3.8), then Theorem 3.1 guarantees the existence of an integral manifold M of the extended system with $M = \{(u, h_0(u)), 0 \leqslant |u| < \infty\}$ where $h_0 : E^\nu \to Q$ is Lipschitz continuous. Furthermore, the proof of that theorem gave h_0 as the limit of the sequence of successive approximations:

$$\begin{aligned} h^0 &= 0, \\ h^{k+1}(u_0) &= -\int_{-\infty}^0 [d_s T(-s) X_0{}^Q]\, G_0{}^e(u^k(s), h^k(u^k(s))) \\ &\quad + \int_{-\infty}^0 T(-s)\, F_0{}^e(u^k(s), h^k(u^k(s)))\, ds, \\ \dot{u}^k(t) &= F_1{}^e(u^k(t), h^k(u^k(t))),\ u^k(0) = u_0 . \end{aligned} \tag{4.14}$$

Furthermore, using the estimate of the Lipschitz constant in u_0, h for $u(t, u_0, h)$ obtained in the proof of Theorem 3.1, we have for every k,

$$|u^k(t, u_0, h^k)| \leqslant e^{-2\nu(\rho)t} |u_0|, \qquad t \geqslant 0. \tag{4.15}$$

Suppose the power series expansion of $F(u, 0)$ beings with terms of degree

$m \geqslant 2$ and ρ_0 in (3.11) is further restricted so that $4(m+1)\,\nu(\rho_0) < \alpha$. From (4.14)

$$h^1(u_0) = \int_{-\infty}^{0} T(-s)\, F_0{}^e(u^0(s), 0)\, ds.$$

From the definition of $F_0{}^e$ in (3.8) and (4.12) and Lemma 4.2, it follows that there is a constant k such that

$$| F_0{}^e(u, 0)| \leqslant k \mid u \mid^{m+1}.$$

Therefore,

$$\begin{aligned} | h^1(u_0)| &\leqslant \int_{-\infty}^{0} Ke^{\alpha s} k e^{-2(m+1)\nu(\rho) s} \mid u_0 \mid^{m+1} ds \\ &\leqslant 2kK\alpha^{-1} \mid u_0 \mid^{m+1}. \end{aligned}$$

A simple induction argument on the sequence h^k allows one to conclude that $h_0(u) = \mathcal{O}(| u |^{m+1})$ as $| u | \to 0$. This is summarized in

LEMMA 4.3. *Suppose the hypotheses of Theorem* 4.1 *are satisfied and* $h_0 : E^\nu \to Q$ *is the Lipschitz continuous function assured by Theorem* 3.1. *If the power series expansion of* $\hat{F}(u, 0)$ *begins with terms of degree* m, *then*

$$h_0(u) = \mathcal{O}(| u |^{m+1}) \qquad \text{as} \qquad | u | \to 0.$$

With h_0 as in Lemma 4.3, the analog of system (3.17) is

$$\dot{u} = \Psi_0 \hat{F}(u, h_0(u)). \tag{4.16}$$

THEOREM 4.2. *Under the assumptions and notations of Theorem* 4.1, *let* $Q(u)$ *designate the homogeneous polynomial of the lowest degree terms in the power series expansion of* $\Psi(0)\hat{F}(u, 0)$. *If the zero solution of the ordinary differential equation*

$$\dot{u} = Q(u) \tag{4.17}$$

is asymptotically stable, then the zero solution of (2.3) *is asymptotically stable* (*and therefore, the degree of* $Q(y)$ *is odd*). *If there is a homogeneous polynomial* $A(u)$ *which is positive on some set and*

$$B(u) = -[\partial A(u)/\partial u]\, Q(u)$$

is negative definite, then the zero solution of (2.3) *is unstable.*

Proof. Suppose the degree of $Q(y)$ is m. If the zero solution of (4.17) is asymptotically stable, it is known from ordinary differential equations [13] that there are two positive definite quadratic forms $A(y)$, $B(y)$ homogeneous of degree $m+1$, $2m$, respectively, such that

$$\dot{A}_{(4.17)}(u) = -B(u),$$

where $\dot{A}_{(4.17)}(u)$ represents the derivative of $A(u)$ along the solutions of (4.17). There is a $\rho_2 > 0$ such that, for $|u| \leqslant \rho_2$, $F_1{}^e(u, h_0(u)) = F_1(u, h_0(u)) = \Psi(0)\hat{F}(u, h_0(u))$. Therefore, for $|u| \leqslant \rho_2$,

$$\dot{A}_{(4.16)}(u) = -B(u) + \frac{\partial A(u)}{\partial u}[\Psi_0\hat{F}(u, h_0(u)) - Q(u)].$$

Lemma 3.3 implies the second term in this expression is at least order $2m+1$ and, therefore, $\dot{A}_{(4.16)}(u)$ is negative definite in a neighborhood of $u = 0$. The classical Liapunov theorem implies that the zero solution of (4.16) is uniformly asymptotically stable. Theorem 3.3 implies the zero solution of (2.3) is uniformly asymptotically stable.

If there is a homogeneous polynomial $A(u)$ of degree $\geqslant 1$ which is not of constant sign and $B(u) = -[\partial A(u)/\partial u]\, Q(u)$ is negative definite, then

$$\dot{A}_{(4.16)}(u) = -B(u) + \frac{\partial A(u)}{\partial u}[\Psi_0\hat{F}(u, h_0(u)) - Q(u)].$$

In a sufficiently small neighborhood of $u = 0$, the right side of this expression is a positive definite function. The classical Cetaev theorem implies that the solution $u = 0$ of (4.16) is unstable. Theorem 3.3 implies the solution $x = 0$ of (2.3) is unstable and the proof is complete.

COROLLARY 4.1. *Under the assumptions and notations of Theorem* 4.1, *let $R(u)$ designate the homogeneous polynomial of the lowest degree terms in the power series expansion of $\Psi(0)\, F(H(\Phi u))$. If the zero solution of the ordinary differential equation*

$$\dot{u} = R(u) \tag{4.18}$$

is asymptotically stable, then the zero solution of (2.3) *is asymptotically stable. If there is a homogeneous polynomial which is positive on some set and*

$$S(u) = -[\partial A(u)/\partial u]\, R(u)$$

is negative definite, then the zero solution of (2.3) *is unstable.*

Proof. Let the degree of $R(u)$ be m. From (4.10), $\Psi(0)\hat{F}(u, 0) = F(H(\Phi u + \alpha(u)))$. Furthermore, from Lemma 4.2, the power series expansions of $\alpha(u)$ begins with terms of at least degree 2. This implies that

$$F(H(\Phi u + \alpha(u))) = R(u) + T(u),$$

where the power series expansion of $T(u)$ beings with terms of at least degree $m + 1$. Theorem 4.2 now gives the result.

Corollary 4.1 includes Theorem 3.1 in [5] for retarded functional differential equations.

COROLLARY 4.2. *Suppose the hypotheses of Theorem* 4.1 *are satisfied and zero is a simple root of* (2.7) *and Eq.* (4.17) *is*

$$\dot{u} = au^m, \qquad a \neq 0. \tag{4.19}$$

If $a < 0$ *and* m *is odd, the solution* $x = 0$ *of* (2.3) *is asymptotically stable. Otherwise, the solution* $x = 0$ *of* (2.3) *is unstable.*

Proof. If $a < 0$ and m is odd, $A(u) = u^2/2$, then

$$\dot{A}_{(4.19)}(u) = au^{m+1}$$

is a negative definite function and the solution $u = 0$ of (4.19) is asymptotically stable. Theorem 4.2 implies the solution $x = 0$ of (2.3) is asymptotically stable.

If $a > 0$, m is odd and $A(u) = u^2/2$, then $\dot{A}_{(4.19)} = -B(u)$ is positive definite. Theorem 4.2 implies the solution $x = 0$ of (2.3) is unstable. If m is even and $A(u) = (\operatorname{sgn} a)u$, then $\dot{A}_{(4.19)}(u) = -B(u) = |a| u^m$ is positive definite. Theorem 4.2 implies the solution $x = 0$ of (2.3) is unstable. The proof is complete.

EXAMPLE. As an example, let us consider the two-dimensional system $[x = \operatorname{col}(x_1, x_2)]$

$$\begin{aligned} \frac{d}{dt} x_1(t) &= x_2(t), \\ \frac{d}{dt} [x_2(t) - g(x(t-r))] &= \alpha x_2(t-r) + f(x(t), x(t-r)), \end{aligned} \tag{4.20}$$

where $r > 0$, f, g are analytic in their arguments in a neighborhood of zero and the power series expansions begin with terms of the degree ≥ 2. The associated linear equation is the RFDE

$$\begin{aligned} \frac{d}{dt} x_1(t) &= x_2(t), \\ \frac{d}{dt} x_2(t) &= \alpha x_2(t-r), \end{aligned} \tag{4.21}$$

which has $a_D = -\infty$ and a characteristic equation given by

$$\lambda(\lambda - \alpha e^{-\lambda r}) = 0. \tag{4.22}$$

For $-\pi/2r < \alpha < 0$, Eq. (4.22) has a simple zero root and all other roots have negative real parts. The bases for the constant solutions of (4.21) and its adjoint may be taken as Φ, Ψ, respectively, with

$$\Phi(0) = \mathrm{col}(\alpha^{-1}, 0), \qquad \Psi(0) = (\alpha, -1).$$

Suppose C is decomposed by $\{0\}$ as $P \oplus Q$ where P is the one-dimensional subspace spanned by Φ. The function H in Theorem 4.1 is given by

$$\begin{aligned} H(\psi)(\theta) &= \psi(\theta), \qquad && -r \leqslant \theta < 0 \\ &\psi(0) + G(\psi(-r)), && \theta = 0, \end{aligned} \tag{4.23}$$

wher $G = \mathrm{col}(0, g)$. If $x_t = H(z_t)$, $z_t = \Phi u(t) + w_t$, then the equations for $u(t)$, w_t are

$$\begin{aligned} &\text{(a)} \quad \dot{u}(t) = -f(z(t) + G(z(t-r)), z(t-r)) \\ &\text{(b)} \quad w_t = T(t)w_0 + \int_0^t \{T(t-s)\, X_0{}^Q f(z(s) + G(z(s-r)), z(s-r))\, ds \\ &\qquad\qquad - [d_s T(t-s)\, X_0{}^Q]\, g(z(s) + G(z(s-r)), z(s-r))\}\, ds. \end{aligned} \tag{4.24}$$

For $w_t = 0$, the right side of (4.24a) becomes

$$\begin{aligned} -f(\Phi u + G(\Phi u), \Phi u) &= -f(\alpha^{-1}u, g(\alpha^{-1}u, 0), \alpha^{-1}u, 0) \\ &\overset{\text{def}}{=} au^m + bu^{m+1} + \cdots. \end{aligned} \tag{4.25}$$

If $\alpha(u)$ is the solution of (4.4) for this particular case, then the fact that $\alpha(u)$ begins with second-order terms in u implies that

$$-f(\Phi u + \alpha(u) + G(\Phi u + \alpha(u)), \Phi u + \alpha(u)) = au^m + cu^{m+1} + \cdots,$$

i.e., the lowest-order terms in the expansion of $-f$ are not affected by $\alpha(u)$.

Therefore, an application of Corollary 4.1 implies the following result. If $a < 0$ and m is odd, the solution $x = 0$ of (4.20) is uniformly asymptotically stable. Otherwise, for any $a \neq 0$, the solution $x = 0$ of (4.21) is unstable.

As a particular illustration, if

$$\begin{aligned} f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= b_1\alpha_1^3 + b_2\alpha_1\alpha_2 + b_3\alpha_3^3, \\ g(\beta_1, \beta_2) &= c_1\beta_1^2, \end{aligned}$$

then m, a in (4.25) are given by

$$m = 3, \; a = (-\alpha)^3(b_1 + b_2c_1 + b_3).$$

Since $\alpha < 0$, it follows that the solution of the equation is asymptotically stable if $b_1 + b_2c_1 + b_3 < 0$, and unstable if $b_1 + b_2c_1 + b_3 > 0$.

References

1. J. K. Hale and M. A. Cruz, Asymptotic behavior of neutral functional differential equations, *Arch. Rational Mech. Anal.* **34** (1969), 331–353.
2. S. Lefschetz, The critical case in differential equations, *Bol. Soc. Mat. Mexicana* (1961), 5–18.
3. V. A. Pliss, On conditional stability in critical cases, *Dokl. Akad. Nauk SSSR* **147** (1962), 1292–1293.
4. S. N. Shimanov, On stability in the critical case of a zero root with time lag, *Prikl. Mat. Meh.* **24** (1960), 447–457.
5. V. P. Prokopev and S. N. Shimanov, On stability in the critical case of two zero roots for systems with lags, *Differential'nye Uravnenija* **2** (1966), 453–462.
6. J. K. Hale, "Ordinary Differential Equations," John Wiley & Sons, New York, 1969.
7. J. K. Hale and M. A. Cruz, Existence, uniqueness and continuous dependence for hereditary systems, *Ann. Mat. Pura Appl.* (4) **85** (1970), 63–82.
8. J. K. Hale, Forward and backward continuation for neutral functional differential equations, *J. Differential Equations* **9** (1971), 168–181.
9. J. K. Hale and K. R. Meyer, A class of functional equations of neutral type, *Mem. Amer. Math. Soc.* **76** (1967), 65 pages.
10. M. A. Cruz and J. K. Hale, Exponential estimates and the saddle point property for neutral functional differential equations, *J. Math. Anal. Appl.* To appear in 1971.
11. N. Chafee, The bifurcation of one or more closed orbits from an equilibrium point of an autonomous differential system, *J. Differential Equations* **4** (1968), 661–679.
12. M. A. Cruz and J. K. Hale, Stability of functional differential equations of neutral type, *J. Differential Equations* **7** (1970), 334–355.
13. V. I. Zubov, "Methods of A. M. Lyapunov and Their Application," Noordhoff, Groningen, 1964.